PARTIAL DIFFERENTIAL EQUATIONS

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1. Overview and preliminaries

(1) Let $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ be any holomorphic function, and let $u := \mathrm{Re} f$ and $v = \mathrm{Im} f$ be the real and imaginary parts of f.

Prove that, if we identify $\mathbb{C} \simeq \mathbb{R}^2$, they satisfy $\Delta u = 0$ and $\Delta v = 0$ in $\Omega \subset \mathbb{R}^2$.

(2 points)

(2) Assume $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ solve Maxwell's equations in $\mathbb{R}^3$

$$\partial_t \vec{\mathbf{E}} = \mathrm{curl}\, \vec{\mathbf{B}}$$
$$\partial_t \vec{\mathbf{B}} = -\,\mathrm{curl}\, \vec{\mathbf{E}}$$
$$\mathrm{div} \vec{\mathbf{B}} = \mathrm{div} \vec{\mathbf{E}} = 0.$$

Prove that

$$\partial_{tt} \vec{\mathbf{E}} - \Delta \vec{\mathbf{E}} = 0 \qquad \text{and} \qquad \partial_{tt} \vec{\mathbf{B}} - \Delta \vec{\mathbf{B}} = 0.$$

(2 points)

(3) Prove that, for any radial function $u \in C^2(\mathbb{R}^n)$, we have

$$\Delta u = \partial_{rr} u + \frac{n-1}{r} \partial_r u = r^{1-n} \partial_r \big(r^{n-1} \partial_r u \big),$$

where $r = |x|$.

(3 points)

(4) Let $f \in C^k(\mathbb{R}^n)$. Prove, for $k = 1, 2, ...$, the following Taylor expansion

$$f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + o(|x|^k) \quad \text{as} \quad x \to 0,$$

where the sum is taken over multiindices $\alpha \in \mathbb{N}^n$.

Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$.

(2 points)

(5) Show that for any C^2 function f we have

$$\fint_{B_r(x_\circ)} f - f(x_\circ) = c_n r^2 \Delta f(x_\circ) + o(r^2),$$

where $\fint_E f = \frac{1}{|E|}\int_E f$ is the average of f over the set E, and $c_n > 0$ is a constant.

(3 points)

(6) Show that for any C^1 vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\mathrm{div}\vec{F}(x) = \frac{1}{|B_1|}\lim_{r\to 0}\frac{1}{r}\int_{\partial B_1}\vec{F}(x+r\theta)\cdot\theta\,dS(\theta).$$

(2 points)

(7) Let $\Omega \subset \mathbb{R}^n$ be any bounded smooth domain. Deduce the following integration by parts formulas from the divergence theorem: For any $f, g \in C^1(\overline{\Omega})$ we have

$$\int_\Omega f\,\partial_{x_i} g = -\int_\Omega g\,\partial_{x_i} f + \int_{\partial\Omega} f\,g\,\nu_i,$$

where ν_i is the i-th component of the normal vector ν. In particular, for any $u, w \in C^2(\overline{\Omega})$,

$$\int_\Omega \nabla w\cdot\nabla u = -\int_\Omega w\,\Delta u + \int_{\partial\Omega} w\,\nabla u\cdot\nu.$$

(2 points)

(8) Find the values of $\alpha > 0$ for which the function $|x|^{-\alpha}$ belongs to $H^1(B_1)$.

(2 points)

(9) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Integrate by parts to prove the interpolation inequality

$$\|\nabla u\|^2_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega)}\|D^2 u\|_{L^2(\Omega)}$$

for all $u \in C^\infty_c(\Omega)$.

(2 points)

(10) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Prove the *trace inequality*

$$\int_{\partial\Omega}|u|^2 \le C\left(\int_\Omega |\nabla u|^2 + |u|^2\right)$$

for all $u \in C^\infty(\overline{\Omega})$.

Hint: Use that there exists a smooth vector field $\vec{X}$ such that $\vec{X}\cdot\nu \ge 1$ on $\partial\Omega$. Then, apply the divergence theorem to $\int_{\partial\Omega} u^2\vec{X}\cdot\nu$ to prove the result.

(2 points)

(11) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain.

(i) Use the previous exercise to show that there is a bounded linear operator

$$\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega)$$

such that $\mathrm{Tr}\,u = u|_{\partial\Omega}$ for any $u \in C^\infty(\overline{\Omega})$. We call it the *trace* operator.

(ii) Prove that there does *not* exist a bounded linear operator

$$\mathrm{Tr} : L^2(\Omega) \to L^2(\partial\Omega)$$

such that $\operatorname{Tr} u = u|_{\partial\Omega}$ for any $u \in C^\infty(\overline{\Omega})$.

Note: This means that we cannot talk about the boundary values of a function in $L^2(\Omega)$, however all functions in $H^1(\Omega)$ do have boundary values in $L^2(\partial\Omega)$.

(4 points)

(12) Prove the Poincaré inequality in dimension $n = 1$, that is:

$$\int_a^b u^2 \leq C_{a,b} \int_a^b |u'|^2 \qquad \text{if} \qquad u(a) = u(b) = 0$$

for some constant $C_{a,b}$ depending only on a and b.

(3 points)

(13) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Use Rellich compactness theorem to prove the following Poincaré inequality

$$\|u - \bar{u}_\Omega\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} \qquad \text{for all} \quad u \in H^1(\Omega),$$

where $\bar{u}_\Omega = \frac{1}{|\Omega|}\int_\Omega u$ is the average value of u in Ω, and C depends only on Ω and n.

Note: We do not assume $u = 0$ on $\partial\Omega$ here.

(4 points)

(14) Define $H^k(\Omega)$, for $k \geq 1$, as the closure of $C^\infty(\overline{\Omega})$ under the norm

$$\|u\|_{H^k(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \cdots + \|D^k u\|_{L^2(\Omega)},$$

where $|D^m u|^2 = \sum_{|\alpha|=m} |\partial_\alpha u|^2$ and $\alpha \in \mathbb{N}^n$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

Prove that $H^k(\Omega)$ is a complete normed space, and that any function $u \in H^k(\Omega)$ has weak derivatives up to order k.

(3 points)

(15) Prove that for any $u \in C_c^\infty(\mathbb{R}^n)$, we have

$$\sup_{\mathbb{R}^n} |u| \leq \|D^n u\|_{L^1(\mathbb{R}^n)}$$

where $D^n u$ denotes the n-th derivatives of u.

Hint: Use the Fundamental Theorem of Calculus n times, with respect to the variables x_1, x_2, ..., x_n.

(2 points)

(16) (i) Prove that the sequence of functions $f_k(x) = x^k$ in $\Omega = (0,1)$, $k = 1, 2, \ldots$, is bounded in $L^2(\Omega)$ but not in $H^1(\Omega)$.

(ii) Prove that the sequence of functions $f_k(x) = \frac{1}{k}\sin(kx)$ in $\Omega = (0,1)$, $k = 1, 2, \ldots$, is bounded in $H^1(\Omega)$. Does the sequence converge in $L^2(\Omega)$? Does it converge in $H^1(\Omega)$?

(2 points)

(17) Let $w_k \in C^m(\overline{\Omega})$ be a sequence of functions satisfying

$$\sum_{k=1}^{\infty} \|w_k\|_{C^m(\overline{\Omega})} < \infty.$$

Prove that the function defined by

$$w(x) := \sum_{k=1}^{\infty} w_k(x)$$

is then $C^m(\overline{\Omega})$, and

$$\|w\|_{C^m(\overline{\Omega})} \leq \sum_{k=1}^{\infty} \|w_k\|_{C^m(\overline{\Omega})}.$$

(3 points)

(18) (i) Let $p > n$. Prove that for any $u \in C_c^\infty(\mathbb{R}^n)$, for all $x \in \mathbb{R}^n$ and all $r > 0$ we have

$$\frac{1}{r^n} \int_{B_r(x)} |u(z) - u(x)| dz \leq C_{n,p} \|\nabla u\|_{L^p(B_r(x))} r^{1-n/p},$$

where $C_{n,p}$ depends only on n and p.

(ii) Use this to prove that for any $u \in C_c^\infty(\mathbb{R}^n)$ and $p > n$, we have

$$[u]_{C^{0,\alpha}(\mathbb{R}^n)} \leq \bar{C}_{n,p} \|\nabla u\|_{L^p(\mathbb{R}^n)}, \qquad \text{where} \qquad \alpha = 1 - \frac{n}{p},$$

where $[u]_{C^{0,\alpha}(\mathbb{R}^n)} = \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.

(4 points)